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On elastic beam models for stability analysis of multilayered rubber bearings

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Abstract

The analysis of the instability phenomena that widely affect the behavior of laminated rubber bearings is examined. A Koiter perturbation strategy is followed. On the basis of two different constitutive relations, two different one-dimensional nonlinear models are used: the first one, linear-elastic, derived from the classical beam theory; the second one, nonlinear-hyperelastic, consistent with the framework of three-dimensional finite elasticity. For the two models, emphasis is placed on the influence of the post-buckling behavior in terms of its load-carrying capacity.

Keywords: Multilayered bearings; Structural stability; Post-buckling analysis; Perturbation approach; Beam models

1. Introduction

The behavior and design of multilayered elastomeric bearings, widely used for bridges and seismic base-isolation of structures, are heavily affected by buckling instability phenomena in axial load compression conditions. In fact, despite their geometric form, this effect is connected with the low ratio of transversal and axial stiffness of these structural elements.

The development of an accurate mechanical model of such instability phenomena needs the setting of two fundamental points: the strategy of analysis and the kinematic-constitutive model of the structural elements. With respect to the strategy of analysis, the attention in literature has so far been focused on the determination of the buckling critical load in axial compression, considered a good estimation of the load-carrying capacity of the element, whereas no attention has been paid to the study of the post-critical behavior, implicitly considered as having no influence on the structural response.

Nevertheless, no rational motivation has been put forward to support these convictions. This could only be obtained on the basis of an effective knowledge of the post-critical behavior of the structural element taken into consideration: it is the author's opinion, in fact, that only that knowledge could give a

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comprehensive view of the structural instability phenomena. Therefore, the study of the critical and post-critical behavior of layered elastomeric bearings is the aim of the present paper. To this end, the paper follows the guidelines defined in Salerno and Lanzo (1997) and Salerno (1992), using a Koiter perturbation strategy (Koiter, 1945; Budiansky, 1974; Casciaro et al., 1991, 1992) that synthetically models the critical and post-critical structural behavior by means of some simple scalar coefficients, and at the same time allows us to take into account the geometric and load imperfection sensitivity in the same direct and simple manner.

With respect to the structural model, prevalent references are made in the literature to one-dimensional beams with shear deformability: this model gives a synthetic representation and is motivated by the constructive features of laminated elastomeric bearings. In particular, the beam constitutive model most widely used is derived from the classical linear elasticity. Exceptions to this are the papers of Marzano and coworkers (Marzano, 1994; D'Ambrosio et al., 1995; Castellano, 1995) where a beam constitutive model is proposed, derived from Blatz and Ko's three-dimensional nonlinear hyperelastic relationship (Blatz and Ko, 1962), that more accurately represents the behavior of the rubber.

The framework becomes confused when we review the kinematical beam models suggested for the buckling analysis of the structural elements which are the object of the present paper. In fact, together with nonlinear beam models derived from technical theory (Engesser, 1891; Haringx, 1948; Timoshenko and Gere, 1961), i.e. enriching the linear beam model with 'ad hoc' nonlinear terms based on diversely motivated assumptions, are also present beam models rationally derived from an exact geometric representation (Reissner, 1972; Antman, 1977, 1995). The reader, discouraged by this picture, can consider the points listed here (see Pignataro et al., 1982; Salerno and Lanzo, 1997; Reissner, 1982):

(1) A nonlinear analysis of the problem, accurate up to any asymptotic order, requires a kinematical nonlinear model accurate up to the same order: only kinematical relations, geometrically exact and thus rationally well-founded, meet such condition.
(2) In the critical analysis a low order of accuracy is required. Within the limits of critical analysis, it can be demonstrated that some of the technical beam models suggested are rationally well-founded, since they are derivable up to the requested order from geometrically exact relations. However, the practicability of such models cannot be transferred to different contexts, for example the post-critical behavior analysis, that require a higher level of accuracy.
(3) Several strain measures, all rationally well founded, can be used. At the kinematical level, no single measurement is better or worse than another. The comparison is only possible at other levels, i.e. on their ability to give an effective and rationally exact representation of constitutive relations.

With the aim of studying the critical and post-critical behavior of layered elastomeric bearings, the present work considers two different beam models, both based on the Cosserat kinematical relation, nonlinear and geometrically exact, suggested by Antman (1977, 1995). The first model, widely used in Salerno and Lanzo (1997), Salerno (1992), Pignataro et al. (1982) and Lanzo (1994)), refers to the classical linear elastic constitutive relationship. Instead, the second model refers to a more complex nonlinear elastic constitutive relationship consistent with the framework of three-dimensional finite elasticity. The first beam model is very similar to the one used by Haringx (1948), but enriched with the axial deformability, in addition to the shear and flexural deformability; the second beam model corresponds exactly to the one suggested by Marzano and coworkers (Marzano, 1994; D'Ambrosio et al., 1995; Castellano, 1995), however, whereas these cited papers are restricted to analysis of the critical behavior, the present paper aims to cover the post-critical range as well.

With reference to the linear-elastic model, the work aims to demonstrate that, for the usual design of bearings, the relative post-critical behavior does not affect the goal of determining its load-carrying capability: this justifies, for the linear-elastic model, referring to the critical analysis only. Such conclusions cannot, however, be extended just then stand to the nonlinear-elastic model. The picture determined by the post-critical behavior of the nonlinear-elastic model is, in fact, more diversified. It's influence on the determination of the load-carrying capability of the structural element is strongly related to the choice of its constitutive parameters and, therefore, to the techniques of homogeneization used for bringing back the real geometry of bearings to the homogeneous one of the examined models.

The work is organized as follows. After a brief description of Koiter's perturbation strategy using the formalism suggested by Budiansky (1974), the Cosserat beam model is synthetically described in its kinematic and static aspects. Having completed the constitutive aspects of the linear-elastic model, its critical and post-critical behavior is analyzed, obtaining in closed form the relative coefficients. Following this, the same treatment is extended to the nonlinear-elastic model, where the critical load is obtained by means of a numerical solution of a nonlinear algebraic equation, and the relative post-critical coefficients are obtained in closed form. Finally, after reporting the experimental numerical results of the two model behavior, there are some considerations and comments.

2. The Koiter strategy of analysis

For structures subjected to conservative loads linearly increasing according to a λ parameter ($p[\lambda] = \lambda\hat{p}$), characterized by an energy of deformation $\Phi[u]$ (where u represents the displacement field compatible with the internal and external kinematical constraints of the structure) and a load potential $p[\lambda]u$ linear in u, the equilibrium configurations are characterized by the stationariety of the total potential energy, expressed by the following virtual work equation

$$\Pi'\delta u = \Phi'\delta u - p\delta u \quad \forall\delta u \tag{1}$$

where a prime indicates Fréchet's derivative with respect to the field u.

For varying loads, the solutions of the problem (1) can be represented by (u, λ) values: such "points" describe in a suitable space one or more "curves" called *equilibrium paths* of the structure. We assume that the natural equilibrium path, that is the path that the structure follows starting from its natural rest configuration, is known and that it is regular in the load parameter $u^{\mathrm{f}}[\lambda]$. We call it *fundamental* path.

The intersection of $u^{\mathrm{f}}[\lambda]$ with a second equilibrium path defines a phenomenon of bifurcation. We call *perfect* the structures that exhibit bifurcation phenomena along their natural equilibrium path. The bifurcation configuration $(u_{\mathrm{b}}, \lambda_{\mathrm{b}})$ is defined by the critical condition

$$\Phi''_{\mathrm{b}}\dot{v}_{\mathrm{b}}\delta u = 0 \quad \forall\delta u \tag{2}$$

that is the singularity tangent stiffness operator in the direction $\dot{v}_{\mathrm{b}}$ (suitably normalized $\|\dot{v}_{\mathrm{b}}\| = 1$) called critical or primary buckling mode.

Starting from the bifurcation point, the branching path, also called *post-critical*, is reconstructed by means of a Koiter asymptotic approach (Koiter, 1945) on the basis of the following representation with reference to a suitable curvilinear abscissa ξ (also see Budiansky, 1974; Casciaro et al., 1991, 1992)

$$\lambda^{\mathrm{d}}[\xi] = \lambda_{\mathrm{b}} + \dot{\lambda}_{\mathrm{b}}\xi + \frac{1}{2}\ddot{\lambda}_{\mathrm{b}}\xi^2 \tag{3a}$$

$$u^{\mathrm{d}}[\xi] = u^{\mathrm{f}}[\lambda^{\mathrm{d}}[\xi]] + \xi\dot{v}_{\mathrm{b}} + \frac{1}{2}\xi^2\ddot{v}_{\mathrm{b}} \tag{3b}$$

where $\ddot{v}_{\mathrm{b}}$ is the secondary buckling mode and $(\dot{\lambda}_{\mathrm{b}}, \ddot{\lambda}_{\mathrm{b}})$ represent, respectively, the slope and curvature (in the bifurcation point) of the curve of the post-critical path.

The scalar parameter $\dot{\lambda}_b$ depends on the bifurcation configuration, the buckling mode and the tangent to the fundamental path $\hat{u} = \frac{\partial}{\partial\lambda} u^f[\lambda]$, being defined by the ratio

$$\dot{\lambda}_b = -\frac{1}{2} \frac{\Phi_b''' \dot{v}_b^3}{\Phi_b''' \hat{u}_b \dot{v}_b^2} \tag{4}$$

For particular symmetry conditions of the problem, such as in the present case

$$\dot{\lambda}_b = 0$$

With this condition, the secondary buckling mode is defined by the solution of the problem

$$\Phi_b'' \ddot{v}_b \delta u + \Phi_b''' \dot{v}_b^2 \delta u = 0 \quad \forall \delta u \tag{5}$$

constrained by a suitable orthogonality condition ($\langle \ddot{v}_b, \dot{v}_b \rangle = 0$); at the same time, the curvature scalar parameter $\ddot{\lambda}_b$ is evaluated by means of the simple ratio of scalar quantities

$$\ddot{\lambda}_b = -\frac{\Phi_b'''' \dot{v}_b^4 - 3\Phi_b'' \ddot{v}_b^2}{3\Phi_b''' \hat{u} \dot{v}_b^2} \tag{6}$$

It can be shown that the choice of the abscissa ξ that describes the post-critical path defines both the normalization condition of the primary buckling mode and its orthogonality condition with the secondary mode

$$(\|\dot{v}_b\| = \langle \dot{v}_b, \dot{v}_b \rangle = 1) \quad (\langle \ddot{v}_b, \dot{v}_b \rangle = 0)$$

It can also be shown that, in presence of load or geometric imperfections that alter the ideal scheme of the perfect structure, the relative equilibrium path is reconstructed with sufficient accuracy by preserving the kinematic representation (3b) but opportunely redefining the λ–ξ relationship according to

$$\lambda + \frac{\mu}{\xi} = \lambda_b + \dot{\lambda}_b \xi + \frac{1}{2} \ddot{\lambda}_b \xi^2 \tag{7}$$

where the effects of the imperfections are taken into account in a simple way through the scalar coefficient μ that, for the imperfections of load $\tilde{p}$, is evaluated by

$$\mu = -\frac{\tilde{p}[\lambda] \dot{v}_b}{\Phi_b''' \hat{u} \dot{v}_b^2} \tag{8}$$

3. The Cosserat beam model

3.1. Kinematic relations

We refer to a Cosserat planar beam model, of generic section and length l, whose generic deformed configuration (see Fig. 1)

$$\begin{aligned} x' &= s + u[s] + z \sin\theta[s] \\ y' &= y \\ z' &= w[s] + z \cos\theta[s] \end{aligned} \tag{9}$$

is defined by the position of the line of centroids in the plane (x, z)

$$\mathbf{p_o}[s] = (s + u[s])\mathbf{i} + w[s]\mathbf{k}$$

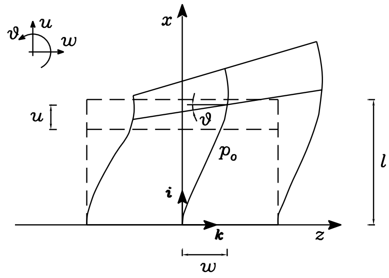

Fig. 1. Kinematics.

and by the rigid rotation $\theta[s]$ of the cross-section, with reference to an initial configuration with rectilinear axis and sections orthogonal to it. Therefore, the kinematics of the model is described in terms of the displacement scalar fields $(u[s], w[s], \theta[s])$.

The strain allowed by the kinematical model includes axial, shear and bending deformations. For finite displacements, exact strain measures $(\varepsilon[s], \gamma[s], \chi[s])$ are defined by the relations see (Salerno and Lanzo, 1997; Antman, 1977, 1995; Pignataro et al., 1982; Lanzo, 1994)

$$\mathbf{r}_{,s} = (1+\varepsilon)\mathbf{a} + \gamma\mathbf{b}, \quad \chi = \theta_{,s}$$

where the unit vectors

$$\mathbf{a} = \cos\theta\mathbf{i} - \sin\theta\mathbf{k}, \quad \mathbf{b} = \sin\theta\mathbf{i} + \cos\theta\mathbf{k}$$

are, respectively, normal and tangent to the plane of the section in the deformed configuration. The development of the above definitions bring to the following nonlinear strain–displacement relationship

$$1+\varepsilon = (1+u_{,s})\cos\theta - w_{,s}\sin\theta \tag{10a}$$

$$\gamma = (1+u_{,s})\sin\theta + w_{,s}\cos\theta \tag{10b}$$

$$\chi = \theta_{,s} \tag{10c}$$

For the structural element under consideration, the kinematic boundary conditions are given instead by (see Fig. 3)

$$\text{for } s=0 : u[0] = w[0] = \theta[0] = 0 \| \quad \text{for } s=l : \theta[l] = 0 \tag{11}$$

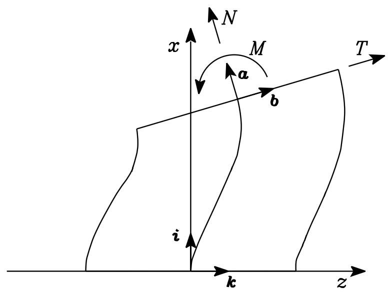

Fig. 2. Internal forces.

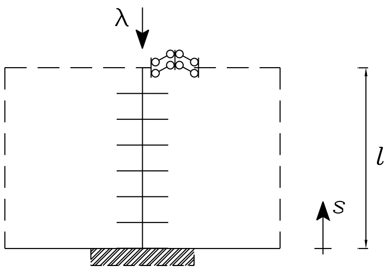

Fig. 3. Beam model.

3.2. Static relations

Let (N, T, M) be force measures such that internal forces and couples of contact action are represented by the vectors (see Fig. 2)

$$\mathbf{t} = N\mathbf{a} + T\mathbf{b}, \quad \mathbf{m} = M\mathbf{b} \times \mathbf{a}$$

In the absence of loads along the axis of the beam, the internal equilibrium conditions

$$\mathbf{t}_{,s} = 0, \quad \mathbf{m}_{,s} + \mathbf{r}_{,s} \times \mathbf{t} = 0$$

are expressed by the three scalar relationships

$$(+N\cos\theta + T\sin\theta)_{,s} = 0 \tag{12a}$$

$$(-N\sin\theta + T\cos\theta)_{,s} = 0 \tag{12b}$$

$$M_{,s} - (1 + u_{,s})(-N\sin\theta + T\cos\theta) + w_{,s}(N\cos\theta + T\sin\theta) = 0 \tag{12c}$$

The equilibrium is completed by the natural boundary conditions. In particular, we consider for the problem an axial compression load λ applied on the top of the structural element (see Fig. 3), and then

$$N[l] = -\lambda, \quad T[l] = 0 \tag{13}$$

Duality can be seen among the characteristics of stress and strain introduced, in the sense that internal virtual work assumes the following expression

$$\int_l \{N\delta\varepsilon + T\delta\gamma + M\delta\chi\}\,\mathrm{d}s$$

where $(\delta\varepsilon, \delta\gamma, \delta\chi)$ are virtual variations of the beam strain parameters. It is therefore natural and rationally correct to define the constitutive model for the beam through relationships in the quantities (N, T, M) and $(\varepsilon, \gamma, \chi)$.

In what follows, we will take into account hyperelastic material relations, for which a strain energy $\Phi[u]$ exists, as a function only of the displacement fields of the structure, such that its first variation is identically equal to the internal virtual work [1]

$$\Phi[u]'\delta u \equiv \int_l \{N\delta\varepsilon + T\delta\gamma + M\delta\chi\}\,\mathrm{d}s \tag{14}$$

[1] The complete expressions of the strain energy variations are reported in Appendix A, together with the expression of the strain parameters variations needed in the following developments of the analysis.

4. The linear-elastic beam model

The first model of multilayered elastomeric bearing is completed in its constitutive representation making use of the classical linear-elastic relationship of the beam theory, i.e.

$$N = EA\varepsilon, \quad T = GA\gamma, \quad M = EJ\chi \tag{15}$$

with EA, GA and EJ axial, shear and flexural stiffness moduli. For this structural model the strain energy is measured by

$$\Phi[u] = \frac{1}{2}\int_l \{EA\varepsilon^2 + GA\gamma^2 + EJ,\chi^2\}\,\mathrm{d}s$$

4.1. Buckling analysis (linear-elastic model)

For varying axial compression loads, the natural fundamental equilibrium path of the bearing is characterized by deformed configurations still rectilinear defined through the following:

$$\begin{matrix} N^{\mathrm{f}}[s] = \mathrm{cost} = N_{\mathrm{o}} = -\lambda \\ T^{\mathrm{f}}[s] = M^{\mathrm{f}}[s] = 0 \end{matrix} \quad \| \quad \begin{matrix} \varepsilon^{\mathrm{f}}[s] = u^{\mathrm{f}}[s]_{,s} = \mathrm{cost} = \varepsilon_{\mathrm{o}} \\ \gamma^{\mathrm{f}}[s] = \chi^{\mathrm{f}}[s] = 0 \end{matrix} \quad \| \quad \begin{matrix} u^{\mathrm{f}}[s] = \varepsilon_{\mathrm{o}} s \\ w^{\mathrm{f}}[s] = \theta^{\mathrm{f}}[s] = 0 \end{matrix} \tag{16}$$

together with the linear relation

$$N = EA\varepsilon_{\mathrm{o}} \tag{17}$$

Along the fundamental path, the critical condition of singularity of the tangent stiffness operator (2) is expressed by

$$\int_l \left\{\dot{N}\delta u_{,s} + \left(\dot{T} - N_{\mathrm{o}}\dot{\theta}\right)\delta w_{,s} + \left(\dot{T}(1+\varepsilon_{\mathrm{o}}) - N_{\mathrm{o}}\left(\dot{w}_{,s} + \dot{\theta}(1+\varepsilon_{\mathrm{o}})\right)\right)\delta\theta + \dot{M}\delta\theta_{,s}\right\}\mathrm{d}s = 0 \quad \forall(\delta u, \delta w, \delta\theta) \tag{18}$$

according to

$$\dot{N} = (N_{\mathrm{o}}'\dot{v}_{\mathrm{b}}), \quad \dot{T} = (T_{\mathrm{o}}'\dot{v}_{\mathrm{b}}), \quad \dot{M} = (M_{\mathrm{o}}'\dot{v}_{\mathrm{b}})$$

Taking into account the boundary conditions (11), it can be demonstrated that such an integral relation is equivalent to the following differential ones:

$$\dot{N} = 0 \quad \| \quad \dot{T} - N_{\mathrm{o}}\dot{\theta} = 0 \quad \| \quad \dot{M}_{,s} + N_{\mathrm{o}}\dot{w}_{,s} = 0 \tag{19}$$

Because of the constitutive relations (15) and

$$\begin{aligned} \dot{N} &= EA(\varepsilon_{\mathrm{o}}'\dot{v}_{\mathrm{b}}) = EA\dot{u}_{,s} \\ \dot{T} &= GA(\gamma_{\mathrm{o}}'\dot{v}_{\mathrm{b}}) = GA\left(\dot{w}_{,s} + \dot{\theta}\left(1 + \frac{N_{\mathrm{o}}}{EA}\right)\right) \\ \dot{M} &= EJ(\chi_{\mathrm{o}}'\dot{v}_{\mathrm{b}}) = EJ\dot{\theta}_{,s} \end{aligned} \tag{20}$$

being valid, the relations (19) are expressed by the differential equation system in the fields $(\dot{u}[s], \dot{w}[s], \dot{\theta}[s])$

$$\dot{u}_{,s} = 0 \quad \| \quad \dot{w}_{,s} = -\left(1 + \frac{N_{\mathrm{o}}}{EA} - \frac{N_{\mathrm{o}}}{GA}\right)\dot{\theta} \quad \| \quad EJ\dot{\theta}_{,ss} - N_{\mathrm{o}}\left(1 + \frac{N_{\mathrm{o}}}{EA} - \frac{N_{\mathrm{o}}}{GA}\right)\dot{\theta} = 0$$

The last equation, with the essential boundary conditions ($\dot{\theta}[0] = \dot{\theta}[l] = 0$), admits the trivial solution $\dot{\theta}[s] = 0$ that agrees with the fundamental path. It can be shown that it also admits solutions in agreement with the form

$$\dot{\theta}[s] = C \sin\left(\frac{n\pi s}{l}\right), \quad n = \{0, 1, 2, \ldots\}$$

for particular critical values N_{c_n} of the internal normal load (and then for particular critical values $\lambda_{c_n} = -N_{c_n}$ of the external load) that fulfill the condition

$$EJ\frac{n^2\pi^2}{l^2} + N_{c_n}\left(1 + \frac{N_{c_n}}{EA} - \frac{N_{c_n}}{GA}\right) = 0$$

These critical values of the compression load correspond to bifurcation conditions along the fundamental path. In particular, we are interested in the lower critical value λ_b: it can be shown that this can be obtained for $n = 1$, and then connected to the critical condition

$$\frac{\pi^2}{l^2}EJ - \lambda_b\left(1 - \frac{\lambda_b}{EA} + \frac{\lambda_b}{GA}\right) = 0 \tag{21}$$

equivalent to an algebraic second degree equation in λ_b

$$\lambda_b^2\left(-\frac{1}{EA} + \frac{1}{GA}\right) + \lambda_b - \frac{\pi^2}{l^2}EJ = 0$$

As $EA \gg GA$, i.e. $(-\frac{1}{EA} + \frac{1}{GA}) > 0$, is the usual design of bearings, the last algebraic equation admits the following real solution:

$$\lambda_b = \frac{-1 + \sqrt{1 + 4\pi^2\frac{EJ}{l^2}\left(-\frac{1}{EA} + \frac{1}{GA}\right)}}{2\left(-\frac{1}{EA} + \frac{1}{GA}\right)} \tag{22}$$

which represents in closed form the critical compression load searched for. To this value the critical mode defined by

$$\dot{u}_{,s} = 0 \quad \| \quad \dot{w}_{,s} = -\left(1 - \frac{\lambda_b}{EA} + \frac{\lambda_b}{GA}\right)C\sin\left(\frac{\pi s}{l}\right) \quad \| \quad \dot{\theta} = C\sin\left(\frac{\pi s}{l}\right)$$

is associated. Taking into account the essential boundary conditions ($\dot{u}[0] = \dot{w}[0] = 0$) and making use of the norm

$$\|\dot{v}_b\| \equiv \dot{w}[l] = 1 \tag{23}$$

the primary critical mode is then expressed by the following displacement fields (see Fig. 4):

$$\dot{u}[s] = 0 \quad \| \quad \dot{w}[s] = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi s}{l}\right) \quad \| \quad \dot{\theta}[s] = -\frac{\pi}{2l}\frac{1}{\left(1 - \frac{\lambda_b}{EA} + \frac{\lambda_b}{GA}\right)}\sin\left(\frac{\pi s}{l}\right) \tag{24}$$

4.1.1. Remarks on the critical behavior of the linear-elastic model

It is interesting to study the relationship (22) to deepen the dependence of the critical load λ_b on the geometric and constitutive parameters of the model. It is possible to rewrite Eq. (22) in this way

$$\lambda_b = \frac{2\lambda_E}{1 + \sqrt{1 + 4\pi^2\left(-\frac{EJ}{EAl^2} + \frac{EJ}{GAl^2}\right)}} \tag{25}$$

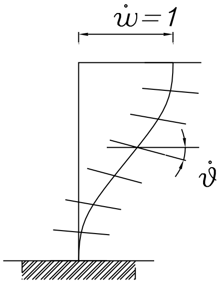

Fig. 4. Buckling model.

as $\lambda_E = \pi^2 \frac{EJ}{l^2}$ (Euler's critical load). This makes it clear that the bifurcation load depends linearly on the normal elasticity modulus E and that axial and transversal rigidities influence the bifurcation load only through the relative ratios with the bending rigidity $\frac{EAl^2}{EJ}$and $\frac{GAl^2}{EJ}$. In particular, at the limit condition $\frac{EAl^2}{EJ} = \infty$ (axially undeformable beam) the expression (25) is coherent with the expression of Haringx's critical load

$$\lambda_H = \frac{2\lambda_E}{1 + \sqrt{1 + 4\pi^2 \frac{EJ}{GAl^2}}},$$

whereas only satisfying both the limit conditions ($\frac{EAl^2}{EJ} = \infty$, $\frac{GAl^2}{EJ} = \infty$) (the *elastical* beam model, axially and tangentially undeformable) the expression (25) coincides identically with Euler's critical load λ_E.

It's worth noting that, As $\Delta = -\frac{EJ}{EAl^2} + \frac{EJ}{GAl^2} > 0$ is the usual design for bearings, it results that $\lambda_b < \lambda_E$. Also it is worth observing that che critical load λ_b decreases for increasing values of the coefficient $\Delta = -\frac{EJ}{EAl^2} + \frac{EJ}{GAl^2}$: as a consequence, λ_b increases for increasing values of the ratio $\frac{GAl^2}{EJ}$ and for decreasing values of the ratio $\frac{EAl^2}{EJ}$. From the last observation, it results that $\lambda_b > \lambda_H$. In conclusion, for the usual bearings design the relation

$$\lambda_H < \lambda_b < \lambda_E$$

is valid.

4.2. Post-critical behavior (linear-elastic model)

Once the fundamental path $u^f[\lambda]$ and, from the resolution of the bifurcation problem, the load and the primary mode of buckling $(\lambda_b, \dot{v}_b)$ are known, the Koiter evaluation of the post-critical path (3) is completed by determining, in sequence, the post-critical slope $\dot{\lambda}_b$, the secondary buckling mode $\ddot{v}_b$ and the post-critical curvature $\ddot{\lambda}_b$. Setting the lateral displacement on the top of the structural element as the curvilinear abscissa of the asymptotic representation of the post-critical path (3).

$$\xi = w[l],$$

it can be demonstrated that it determines the preceding normalization condition (23) of the primary buckling mode and the following orthogonality condition between primary and secondary buckling mode

$$\dot{v}_b \perp \ddot{v}_b \Longleftrightarrow \dot{w}[l]\ddot{w}[l] = \ddot{w}[l] = 0 \quad (26)$$

4.2.1. The post-critical slope

The post-critical slope $\dot{\lambda}_b$ is defined in (4) by means of the ratio of two scalar quantities computed by (see Appendix A)

$$\begin{aligned} &\Phi_{\mathrm{b}}'''\dot{v}_{\mathrm{b}}^3 = 0 \\ &\Phi_{\mathrm{b}}'''\hat{u}_{\mathrm{b}}\dot{v}_{\mathrm{b}}^2 = -\left(1 - 2\frac{\lambda_{\mathrm{b}}}{EA} + 2\frac{\lambda_{\mathrm{b}}}{GA}\right)\int_l \dot{\theta}^2\,\mathrm{d}s \end{aligned} \tag{27}$$

As a result we have

$$\dot{\lambda}_{\mathrm{b}} = -\frac{1}{2}\frac{\Phi_{\mathrm{b}}'''\dot{v}_{\mathrm{b}}^3}{\Phi_{\mathrm{b}}'''\hat{u}_{\mathrm{b}}\dot{v}_{\mathrm{b}}^2} = 0$$

In conclusion and coherently with the symmetry of the problem, the post-critical path results in a zero value of its initial slope (at the bifurcation configuration).

4.2.2. The secondary buckling mode

The problem (5) that determines the secondary buckling mode $\ddot{v}_{\mathrm{b}} \equiv \{\ddot{u}[s], \ddot{w}[s], \ddot{\theta}[s]\}$ is defined by the tangent stiffness operator $\Phi_{\mathrm{b}}''(\cdot)(\cdot)$ and by the following known terms:

$$\Phi_{\mathrm{b}}'''\dot{v}_{\mathrm{b}}^2\delta u = \int_l \left\{EA\left(1 - 2\frac{\lambda_{\mathrm{b}}}{EA} + 2\frac{\lambda_{\mathrm{b}}}{GA}\right)\dot{\theta}^2\delta u_{,s}\right\}\mathrm{d}s$$

The problem (5) is then expressed by

$$\begin{aligned} \int_l \Bigg\{ &\left(\ddot{N} + EA\left(1 - 2\frac{\lambda_{\mathrm{b}}}{EA} + 2\frac{\lambda_{\mathrm{b}}}{GA}\right)\dot{\theta}^2\right)\delta u_{,s} + \left(\ddot{T} - N_{\mathrm{b}}\ddot{\theta}\right)\delta w_{,s} \\ &+ \left(\ddot{T}(1+\varepsilon_{\mathrm{b}}) - N_{\mathrm{b}}\left(\ddot{w}_{,s} + \ddot{\theta}(1+\varepsilon_{\mathrm{b}})\right)\right)\delta\theta + \ddot{M}\delta\theta_{,s}\Bigg\}\mathrm{d}s = 0 \quad \forall(\delta u, \delta w, \delta\theta) \end{aligned}$$

where

$$\ddot{N} = (N_{\mathrm{b}}'\ddot{v}_{\mathrm{b}}) = EA\ddot{u}_{,s}, \quad \ddot{T} = (T_{\mathrm{b}}'\ddot{v}_{\mathrm{b}}) = GA(\ddot{w}_{,s} + (1+\varepsilon_{\mathrm{b}})\ddot{\theta}), \quad \ddot{M} = (M_{\mathrm{b}}'\ddot{v}_{\mathrm{b}}) = EJ\ddot{\theta}_{,s}$$

With developments similar to those carried out in the bifurcation problem and taking into account the orthogonality condition (26), this problem admits the following solution in the components of the secondary buckling mode:

$$\ddot{u}_{,s} = -\left(1 - 2\frac{\lambda_{\mathrm{b}}}{EA} + 2\frac{\lambda_{\mathrm{b}}}{GA}\right)\dot{\theta}^2, \quad \ddot{w}[s] = \ddot{\theta}[s] = 0$$

4.2.3. The post-critical curvature

Once the secondary buckling mode is known, the post-critical curvature parameter $\ddot{\lambda}_{\mathrm{b}}$ is computed by the ratio (6) with a denominator given by (27) and a numerator given by the following terms:

$$\Phi_{\mathrm{b}}''''\dot{v}_{\mathrm{b}}^4 = 3EA\left(1 - 2\frac{\lambda_{\mathrm{b}}}{EA} + 2\frac{\lambda_{\mathrm{b}}}{GA}\right)^2\int_l \dot{\theta}^4\,\mathrm{d}s + \lambda_{\mathrm{b}}\left(1 - 4\frac{\lambda_{\mathrm{b}}}{EA} + 4\frac{\lambda_{\mathrm{b}}}{GA}\right)\int_l \dot{\theta}^4\,\mathrm{d}s$$

$$\Phi_{\mathrm{b}}''\ddot{v}_{\mathrm{b}}^2 = \int_l \left\{\ddot{N}(\varepsilon_{\mathrm{b}}'\ddot{v}_{\mathrm{b}})\right\}\mathrm{d}s = EA\left(1 - 2\frac{\lambda_{\mathrm{b}}}{EA} + 2\frac{\lambda_{\mathrm{b}}}{GA}\right)^2\int_l \dot{\theta}^4\,\mathrm{d}s$$

From this we obtain

$$\ddot{\lambda}_{\mathrm{b}} = -\frac{\Phi_{\mathrm{b}}''''\dot{v}_{\mathrm{b}}^4 - 3\Phi_{\mathrm{b}}''\ddot{v}_{\mathrm{b}}^2}{3\Phi_{\mathrm{b}}'''\hat{u}\dot{v}_{\mathrm{b}}^2} = \lambda_{\mathrm{b}}\frac{\left(1 - 4\frac{\lambda_{\mathrm{b}}}{EA} + 4\frac{\lambda_{\mathrm{b}}}{GA}\right)}{\left(1 - 2\frac{\lambda_{\mathrm{b}}}{EA} + 2\frac{\lambda_{\mathrm{b}}}{GA}\right)}\frac{\left(\frac{\pi}{4l}\right)^2}{\left(1 - \frac{\lambda_{\mathrm{b}}}{EA} + \frac{\lambda_{\mathrm{b}}}{GA}\right)^2} \tag{28}$$

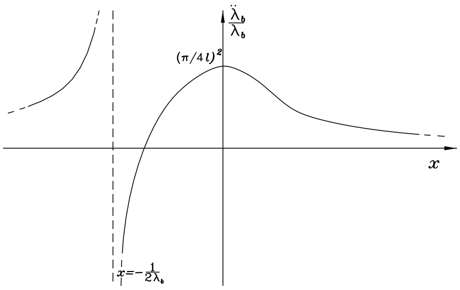

Fig. 5. Values $\frac{\ddot{\lambda}_b}{\lambda_b}$ vs. $x = -\frac{1}{EA} + \frac{1}{GA}$.

It is a simple task to demonstrate that the post-critical curvature coefficient depends linearly on the normal elasticity modulus.

4.2.4. Remarks on the post-critical behavior of the linear-elastic model

A measure of the influence of the post-critical behavior is given by the relationship between the initial curvature of the post-critical path and the bifurcation load value $\left(\frac{\ddot{\lambda}_b}{\lambda_b}\right)$. For Eq. (28), this ratio is evaluated by

$$\frac{\ddot{\lambda}_b}{\lambda_b} = \left(\frac{\pi}{4l}\right)^2 \frac{\left(1 + 4\pi^2 \frac{\lambda_b}{\lambda_E} \Delta\right)}{\left(1 + 2\pi^2 \frac{\lambda_b}{\lambda_E} \Delta\right)\left(1 + \pi^2 \frac{\lambda_b}{\lambda_E} \Delta\right)^2}$$

in terms of the coefficient $\Delta = -\frac{EJ}{EAl^2} + \frac{EJ}{GAl^2}$ and of the ratio $\frac{\lambda_b}{\lambda_E}$ between actual and Eulerian bifurcation load. For positive range values of the coefficient Δ (the only interesting one from a technical point of view) and for several values of $\frac{\lambda_b}{\lambda_E} < 1$, its values are represented qualitatively in the graph of Fig. 5. It can be observed that the post-critical coefficient $\frac{\ddot{\lambda}_b}{\lambda_b}$ always assumes positive values which are always less than $\left(\frac{\pi}{4l}\right)^2$, i.e. the limit value of the coefficient relative to the *elastical* model ($\frac{\lambda_b}{\lambda_E} = 1$). In particular, for increasing values of the coefficient Δ the values of $\frac{\ddot{\lambda}_b}{\lambda_b}$ are rapidly decreasing and, in the cases of technical interest, practically became $\frac{\ddot{\lambda}_b}{\lambda_b} \approx 0$. In conclusion, positive values of the ratio $\frac{\ddot{\lambda}_b}{\lambda_b}$ have a stabilizing effect of the post-critical behavior on the general limit behavior of the model. However, this effect is very limited because of the low value of this ratio. This conclusion is also validated by the numerical results obtained from the examples reported further on.

5. The nonlinear-elastic beam model

The new beam model is obtained for the same kinematical (9) and (10), and then static (12) and (13), relationships, but enriching the constitutive relations to better take into account the nonlinear elastic behavior of rubber materials. In particular we refer to isotropic, hyperelastic constitutive relations, opportunely deduced in Marzano (1994), D'Ambrosio et al. (1995) from well-known Blatz and Ko equations (Blatz and Ko, 1962), characterized by the following volume density of the strain energy [2]

[2] The case $\rho = 0$ agrees exactly with the Blatz and Ko constitutive model.

$$\varphi[\mathbf{F}] = \frac{1}{2}\mu\left[\|\mathbf{F}\|^2 - 3 + \frac{2}{\sigma}((\det\mathbf{F})^{-\sigma} - 1)\right] + \frac{1}{4}\rho(|\mathbf{Fi}|^2 - 1)^2$$

where the tensor $\mathbf{F}$ is the gradient of the deformation (9) and (μ, σ, ρ) are the constitutive coefficients of the rubber. Recalling the definitions (10) we obtain

$$\varphi[u, w, \theta] = \frac{1}{2}\mu\left[-1 + (1 + \varepsilon + z\chi)^2 + \gamma^2 + \frac{2}{\sigma}((1 + \varepsilon + z\chi)^{-\sigma} - 1)\right] + \frac{1}{4}\rho\left[(1 + \varepsilon + z\chi)^2 + \gamma^2 - 1\right]^2$$

The strain energy is then computed by

$$\Phi[u, w, \theta] = \int_l \left(\int_A \varphi[u, w, \theta]\mathrm{d}A\right)\mathrm{d}s$$

On the basis of the identity

$$\Phi'\delta u = \int_l \left(\int_A (\varphi'\delta u)\,\mathrm{d}A\right)\mathrm{d}s = \int_l (N\delta\varepsilon + T\delta\gamma + M\delta\chi)\mathrm{d}s$$

we get the following representation of the constitutive relations for the equivalent beam model:

$$N = \int_A \frac{\partial}{\partial\varepsilon}\varphi[\varepsilon, \gamma, \chi]\mathrm{d}A \tag{29a}$$

$$T = \int_A \frac{\partial}{\partial\gamma}\varphi[\varepsilon, \gamma, \chi]\mathrm{d}A \tag{29b}$$

$$M = \int_A \frac{\partial}{\partial\chi}\varphi[\varepsilon, \gamma, \chi]\mathrm{d}A \tag{29c}$$

Explicit expressions of these constitutive relations and other variational details are given in Appendix A.

5.1. Critical behavior (nonlinear-elastic model)

The structural model still exhibits the trivial fundamental path (16) where, however, because of the constitutive equation (29a), the following nonlinear relation between the normal stress and axial shortening parameter $k = 1 + \epsilon_o$ is valid

$$N_o = \mu A(k - k^{-(1+\sigma)}) + \rho A(k(k^2 - 1)) \tag{30}$$

Here A stands for the area of the beam cross-section while, in the following pages:

$$J \stackrel{\text{def}}{=} \int_A z^2\mathrm{d}A, \quad J_4 \stackrel{\text{def}}{=} \int_A z^4\,\mathrm{d}A$$

(It is supposed that $\int_A z\mathrm{d}A = 0$ and $\int_A z^3\mathrm{d}A = 0$.)

The bifurcation condition (2) is still expressed by Eq. (18) and then by Eqs. (19). Because of the new constitutive model, these now give the following nonlinear differential equation system in the fields $(\dot{u}[s], \dot{w}[s], \dot{\theta}[s])$ (the critical mode):

$$\dot{u}_{,s} = 0 \quad \| \quad \dot{w}_{,s} = -\frac{\eta[k]}{\beta[k]}\dot{\theta} \quad \| \quad J\alpha[k]\dot{\theta}_{,ss} - N_o[k]\frac{\eta[k]}{\beta[k]}\dot{\theta} = 0 \tag{31}$$

being set (see Appendix A for expressions of functions $\alpha[k]$, $\beta[k]$ and $\eta[k]$)

$$\begin{array}{lcl} N_o = Ak\beta[k] - A\eta[k] & \| & \dot{T} = (T_o'\dot{v}_b) = A\beta[k](\dot{w}_{,s} + k\dot{\theta}) \\ \dot{N} = (N_o'\dot{v}_b) = A\alpha[k]\dot{u}_{,s} & & \dot{M} = (T_o'\dot{v}_b) = J\alpha[k]\dot{\theta}_{,s} \end{array} \tag{32}$$

Satisfying the boundary conditions and the normalization (23), the system (31) admits a solution in the form

$$\dot{u}[s] = 0 \quad \| \quad \dot{w}[s] = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi s}{l}\right) \quad \| \quad \dot{\theta}[s] = -\frac{\pi}{2l}\frac{\beta[k_b]}{\eta[k_b]}\sin\left(\frac{\pi s}{l}\right)$$

in relation to the smaller k_b value that realizes the condition

$$J\alpha[k_b]\frac{\pi^2}{l^2} + N_o[k_b]\frac{\eta[k_b]}{\beta[k_b]} = 0 \tag{33}$$

and then in connection with the bifurcation load

$$\lambda_b = -N_o[k_b]$$

It is worth observing that, in contrast to the solution of problem (21), the solution of the bifurcation problem (33) cannot be obtained in closed form. That solution, however, can be computed by means of common procedures of numerical analysis. Also in this case, it can be demonstrated that the critical value, for the same values of the constitutive coefficient σ and of the ratio ρ/μ, depends linearly on the elasticity modulus μ.

5.2. Post-critical behavior (nonlinear-elastic model)

5.2.1. The post-critical slope

As

$$\Phi_b'''\dot{v}_b^3 = 0$$

the value of the post-critical initial slope is given again by

$$\dot{\lambda}_b = -\frac{1}{2}\frac{\Phi_b'''\dot{v}_b^3}{\Phi_b'''\hat{u}_b\dot{v}_b^2} = 0$$

5.2.2. The secondary buckling mode

The known terms of the secondary buckling mode problem (5) are now given by

$$\Phi_b'''\dot{v}_b^2\delta u = \int_l \left\{ A\left(\alpha_b k_b + (\beta_b - 2\alpha_b)\frac{N_b}{A\beta_b} + 2\rho k_b\frac{N_b^2}{A^2\beta_b^2}\right)\dot{\theta}^2\delta u_{,s} + J\omega_b\dot{\theta}_{,s}^2\delta u_{,s}\right\}\mathrm{d}s$$

with

$$N_b = N_o[k_b], \quad \alpha_b = \alpha[k_b], \quad \beta_b = \beta[k_b], \quad \eta_b = \eta[k_b], \quad \omega_b = \frac{\partial\alpha}{\partial x}[k_b]$$

The problem (5) is then expressed by

$$\int_l \left\{ \left(\ddot{N} + A\left(\alpha_b k_b + (\beta_b - 2\alpha_b)\frac{N_b}{A\beta_b} + 2\rho k_b\frac{N_b^2}{A^2\beta_b^2}\right)\dot{\theta}^2 + J\omega_b\dot{\theta}_{,s}^2\right)\delta u_{,s} + \left(\ddot{T} - N_b\ddot{\theta}\right)\delta w_{,s} \right.$$
$$\left. + \left(\ddot{T}k_b - N_b\left(\ddot{w}_{,s} + \ddot{\theta}k_b\right)\right)\delta\theta + \ddot{M}\delta\theta_{,s}\right\}\mathrm{d}s = 0, \quad \forall(\delta u, \delta w, \delta\theta)$$

where

$$\ddot{N} = (N_{\rm b}'\ddot{v}_{\rm b}) = A\alpha_{\rm b}\ddot{u}_{,s}, \quad \ddot{T} = (T_{\rm b}'\ddot{v}_{\rm b}) = A\beta_{\rm b}(\ddot{w}_{,s} + k_{\rm b}\ddot{\theta}), \quad \ddot{M} = (M_{\rm b}'\ddot{v}_{\rm b}) = J\alpha_{\rm b}\ddot{\theta}_{,s}$$

Taking into account the orthogonalization condition (26), the problem admits the following solution in the components $\ddot{v}_{\rm b} \equiv (\ddot{u}, \ddot{w}, \ddot{\theta})$ of the secondary buckling mode

$$\ddot{u}_{,s} = -\left(k_{\rm b} + \left(\frac{\beta_{\rm b}}{\alpha_{\rm b}} - 2\right)\frac{N_{\rm b}}{A\beta_{\rm b}} + 2\frac{\rho k_{\rm b}}{\alpha_{\rm b}}\frac{N_{\rm b}^2}{A^2\beta_{\rm b}^2}\right)\dot{\theta}^2 - \frac{J\omega_{\rm b}}{A\alpha_{\rm b}}\dot{\theta}_{,s}^2, \quad \ddot{w}[s] = \ddot{\theta}[s] = 0$$

5.2.3. The post-critical curvature

The developments (reported in Appendix A) of the several terms lead to the following expression for the post-critical initial curvature coefficient

$$\ddot{\lambda}_{\rm b} = -\frac{\Phi_{\rm b}''''\dot{v}_{\rm b}^4 - 3\Phi_{\rm b}''\ddot{v}_{\rm b}^2}{3\Phi_{\rm b}'''\hat{u}\dot{v}_{\rm b}^2} = \frac{1}{4}\left(\frac{\pi}{2l}\frac{\beta_{\rm b}}{\eta_{\rm b}}\right)^2 A\frac{\left(C_0 + \frac{N_{\rm b}}{A\beta_{\rm b}}C_1 + \frac{N_{\rm b}^2}{A^2\beta_{\rm b}^2}C_2 + \frac{N_{\rm b}^3}{A^3\beta_{\rm b}^3}C_3 + \frac{N_{\rm b}^4}{A^4\beta_{\rm b}^4}C_4\right)}{D_0 + \frac{N_{\rm b}}{A\beta_{\rm b}}D_1 + \frac{N_{\rm b}^2}{A^2\beta_{\rm b}^2}D_2}$$

where the parameters $(C_0, C_1, C_2, C_3, C_4, D_0, D_1, D_2)$ are connected to the constitutive parameters of the model and to the coefficient $k_{\rm b}$: their expression is given in Appendix A. It can also be demonstrated that, for the same values of σ and ρ/μ, the post-critical curvature $\ddot{\lambda}_{\rm b}$ depends linearly on the elasticity modulus μ.

6. A condition of critical equivalence between models

It's worth noting that the two beam models examined are quite different. In fact, going beyond the different models of elastic isotropic material which the two models refer to (linear-elastic and nonlinear-elastic), it's sufficient to observe that the linear-elastic model is defined in the context of first order theory (on the basis of the St. Venant hypothesis and by means of 'ad hoc' technical assumptions for the shear deformability) and projected in a geometrically nonlinear context. The nonlinear-elastic model, instead, is directly defined in a nonlinear geometrically exact context and coherently reduced to a nonlinear-elastic beam on the basis of a precise kinematical constraint (of cross-section rigidity).

However, with the aim of comparing in quantitative terms their post-critical behavior, it is possible to define the constitutive coefficients (EA, GA, EJ) of the linear-elastic model by means of a condition of equivalence, in terms of critical load, with the nonlinear model, i.e. starting from the resolution of the critical problem of the nonlinear model and obtaining the following relations:

$$EA_{\rm eq} = \frac{N_{\rm b}}{k_{\rm b} - 1}, \quad GA_{\rm eq} = A \cdot \beta_{\rm b}, \quad EJ_{\rm eq} = J \cdot \alpha_{\rm b} \tag{34}$$

from (17), (19), (20) and (32).

7. Numerical results

The aim of the numerical experimentation is to study, from a qualitative and quantitative point of view, the influence, in terms of load-carrying capacity, of the post-critical behavior of the two beam models of laminated bearings proposed. In this section some of the results obtained for ideal examples of homogenized bearings are reported having

- the same sectional geometry (circular and hollow, with external radius $R_{\text{ext}} = 120$ mm and internal radius $R_{\text{int}} = 10$ mm);
- several values of height l, and then several slenderness values $sl = l/\sqrt{\frac{J}{A}}$ as reported in the following:

l (mm)	31	61	121	133	181	241	301
sl	0.515	1.013	2.009	2.209	3.006	4.003	4.999

- several values of the constitutive coefficients (EA, GA, EJ) of the linear model on the basis of

$$EA = E \cdot A, \quad GA = G \cdot A, \quad EJ = E \cdot J, \quad G = \frac{E}{2(1+v)}$$

- several values of the constitutive coefficients (μ, σ, ρ) of the nonlinear model and, then, of the constitutive coefficients $(EA_{\text{eq}}, GA_{\text{eq}}, EJ_{\text{eq}})$ of the equivalent linear model.

In the following pages, imperfect structural elements will be obtained taking into account small additional imperfectional loads defined by a horizontal force $\tilde{p}[\lambda] = T$ applied to the top of the elements. From Eqs. (7), (8) and (23) and with the post-critical slope coefficient $\dot{\lambda}_{\text{b}}$ being zero for the cases taken into account, the λ–ξ relationship is given in implicit form by

Table 1
Linear-elastic model: critical and post-critical values for varying values of the constitutive coefficient σ and of the slenderness ($G = 0.4407$ MPa)

v	l (mm)	λ_{b} (N)	λ_{H} (N)	$\ddot{\lambda}_{\text{b}}$ (MPa)	$\frac{1}{2} \times \frac{\ddot{\lambda}_{\text{b}}}{\lambda_{\text{b}}} \times l^2$
0.4545	31	239702.740	(196377.662)	3.731762393	0.007480564
	61	115047.662	(95276.151)	1.550904150	0.025080537
	133	46086.895	(39134.634)	0.441072514	0.088464588
	181	31014.407	(26751.477)	0.237507994	0.125441693
	241	20941.582	(18392.180)	0.122536612	0.169926248
	301	15142.905	(13516.232)	0.068599887	0.205218830
0.4762	31	240716.458	(197903.284)	3.670534423	0.007326843
	61	115607.287	(96048.934)	1.528870980	0.024604543
	133	46378.632	(39483.359)	0.437059881	0.083348213
	181	31238.838	(27003.416)	0.236095988	0.123800069
	241	21114.699	(18576.310)	0.122242150	0.168128049
	301	15281.813	(13658.881)	0.068643984	0.203484158
0.4950	31	241605.049	(199223.465)	3.619327473	0.007198056
	61	116096.044	(96717.684)	1.510390500	0.024204800
	133	46632.142	(39785.203)	0.433668053	0.082251780
	181	31433.540	(27221.533)	0.234894803	0.122407286
	241	21264.732	(18735.776)	0.121987954	0.166594676
	301	15402.161	(13782.468)	0.068679234	0.201997870
0.5000	31	241839.063	(199568.634)	3.606200579	0.007165010
	61	116224.499	(96892.537)	1.505645111	0.024102084
	133	46698.582	(39864.134)	0.432793201	0.081969072
	181	31484.520	(27278.576)	0.234583838	0.122047298
	241	21303.993	(18777.488)	0.121921586	0.166197189
	301	15433.648	(13814.803)	0.068688011	0.201611519

$$\lambda - \frac{T}{\xi \Phi_{\mathrm{b}}''' \hat{u} \dot{v}_{\mathrm{b}}^2} = \lambda_{\mathrm{b}} + \frac{1}{2}\ddot{\lambda}_{\mathrm{b}}\xi^2 \tag{35}$$

This relation is determined by the knowledge of the three scalar coefficients ($\lambda_{\mathrm{b}}, \ddot{\lambda}_{\mathrm{b}}, \Phi_{\mathrm{b}}''' \hat{u} \dot{v}_{\mathrm{b}}^2$) only. Following the perturbation approach, for given values of the horizontal load T, this equation defines, in the manifold (3b), the equilibrium path of the imperfect structural elements, i.e. the values of the axial load λ for varying values of the horizontal displacement ξ in the top of the bearing.

7.1. Results of the linear-elastic model

The results obtained for the linear-elastic model in terms of critical and post-critical behavior are summarized in Table 1. As can be observed, the critical value λ_{b} is obviously strongly affected by the slenderness values, but much less by the coefficient values v. For comparison, Haringx's critical values are also reported in the table: as already observed, $\lambda_{\mathrm{b}} > \lambda_{\mathrm{H}}$ is always obtained, with $10 \approx 15\%$ difference in values for the range of cases tested.

The numerical results of Table 1 verify what has already been observed regarding the qualitative post-critical behavior of the model. Low positive values of the post-critical curvature coefficient $\ddot{\lambda}_{\mathrm{b}}$ indicate a stabilizing effect, albeit small, on the general behavior of the structural model. In fact, for a transversal post-buckling displacement on the top of the bearing equal to 100% of the relative height, the load

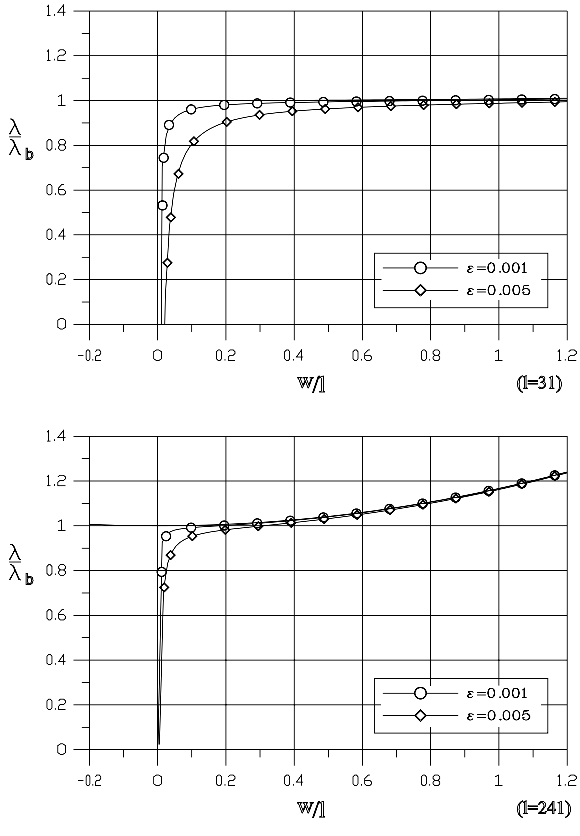

Fig. 6. Equilibrium paths of the linear-elastic model.

increment along the post-buckling path (measured by the coefficient $\Delta_{pc} = \frac{1}{2} \times \frac{\ddot{\lambda}_b}{\lambda_b} \times l^2$ reported in the table) reaches the maximum value of 20% of the bifurcation load only for the more slender bearings (and then for the relatively greater transversal displacements relatively). It is also worth noting that this post-critical coefficient is substantially unaffected by the elastic constitutive parameters (G, v).

The curves of the equilibrium paths, typical of the observed behavior, are reconstructed in the diagrams of Fig. 6. With reference to the perfect structural element (absence of imperfections), the path bifurcates with a substantially flat post-buckling path. For several values of load imperfections, the curves of the equilibrium paths are asymptotic to the two paths of the bifurcation phenomenon, and then express a load-carrying capacity of the structural element practically by the value λ_b of the critical load.

7.2. Results of the nonlinear-elastic model

The results obtained for the nonlinear-elastic model in terms of critical and post-critical behavior are reported in Tables 2 and 3. The critical load values are obviously strongly connected to the slenderness of the bearings. In contrast to what observed for the linear-elastic model, the dependence of the critical value on the constitutive coefficient σ (increasing values for decreasing values of σ) is greater. Instead the dependence of the critical values on the ratio ρ/μ, nevertheless with a variation up to 40% for low values of σ, is more limited.

The general view of the post-critical behavior of the nonlinear-elastic model appears quite variegated. As summarized in the diagram of Fig. 7, the relative parameter $\Delta_{pc} = \frac{1}{2} \times \frac{\ddot{\lambda}_b}{\lambda_b} \times l^2$ is quite variable in relation to the coefficient σ (however, this variability decreases for decreasing slenderness). From positive to negative values, this post-critical parameter is strongly influenced by the ratio ρ/μ. The case $\rho/\mu = 0$, corresponding exactly to Blatz and Ko's rubber elasticity model, is relevant: negative values of the post-critical coefficient Δ_{pc} can be noticed, that negatively affect the load-carrying capacity of the structural element. For increasing

Table 2
Nonlinear-elastic model: critical and post-critical values for varying values of the constitutive parameter σ and of the ratio ρ/μ (height $l = 133$ mm, $\mu = 0.4407$ MPa)

σ	ρ/μ	k_b	λ_b (N)	$\ddot{\lambda}_b$ (MPa)	$\frac{1}{2} \times \frac{\ddot{\lambda}_b}{\lambda_b} \times l^2$	$(\frac{1}{2} \times \frac{\ddot{\lambda}_b}{\lambda_b} \times l^2)_{eq}$
10	0.0	0.731388622	603534.356	−10.769	−0.1578	(0.1040×10^{-8})
	0.1	0.734806532	573239.978	2.804	0.0433	(0.1210×10^{-8})
	0.5	0.749492629	460680.260	51.946	0.9973	(0.2091×10^{-8})
	0.8	0.761560921	386155.295	82.097	1.8804	(0.2382×10^{-8})
	1.0	0.770062881	341582.565	98.312	2.5456	(0.2495×10^{-8})
20	0.0	0.828380048	1015901.349	−17.911	−0.1559	(0.8027×10^{-10})
	0.1	0.829685760	982802.838	4.104	0.0369	(0.2451×10^{-10})
	0.5	0.835163061	855791.572	87.849	0.9079	(0.3467×10^{-10})
	0.8	0.839545916	766586.664	145.284	1.6762	(0.3706×10^{-9})
	1.0	0.842600440	710216.716	180.516	2.2489	(0.3779×10^{-9})
30	0.0	0.870786313	1426102.822	−25.053	−0.1554	(0.4135×10^{-10})
	0.1	0.871490920	1390763.578	5.492	0.0349	(0.8738×10^{-10})
	0.5	0.874415928	1253455.300	123.647	0.8725	(0.1134×10^{-9})
	0.8	0.876725439	1154969.526	207.398	1.5882	(0.1185×10^{-9})
	1.0	0.878321927	1091588.093	260.539	2.1110	(0.1198×10^{-9})
100	0.0	0.948153131	4265747.695	−74.632	−0.1547	(0.1845×10^{-11})
	0.1	0.948249134	4222344.713	15.316	0.0321	(0.3180×10^{-11})
	0.5	0.948639577	4050372.194	371.516	0.8112	(0.3529×10^{-11})
	0.8	0.948939326	3923160.135	634.636	1.4307	(0.3577×10^{-11})
	1.0	0.949142540	3839218.903	807.988	1.8614	(0.3583×10^{-11})

Table 3
Nonlinear-elastic model: critical and post-critical values for varying values of the constitutive coefficient σ and of the slenderness ($\rho/\mu = 0.8$, $\mu = 0.4407$ MPa)

σ	l (mm)	k_b	λ_b (N)	$\ddot{\lambda}_b$ (MPa)	$\frac{1}{2}\times\frac{\ddot{\lambda}_b}{\lambda_b}\times l^2$	$\left(\frac{1}{2}\times\frac{\ddot{\lambda}_b}{\lambda_b}\times l^2\right)_{eq}$
10	31	0.589865885	6576186.377	41531.918	3.0346	(0.1941×10^{-12})
	61	0.665260632	1745448.125	2389.857	2.5472	(0.1029×10^{-10})
	121	0.749422096	463122.549	124.997	1.9758	(0.7118×10^{-9})
	181	0.801576171	214215.138	19.922	1.5234	(0.9629×10^{-8})
	241	0.838634154	124530.043	4.821	1.1242	(0.5128×10^{-7})
	301	0.866475189	82053.573	1.426	0.7876	(0.1797×10^{-6})
20	31	0.733415674	13303819.850	63581.898	2.2964	(0.6916×10^{-13})
	61	0.781233631	3523052.456	3904.147	2.0617	(0.2180×10^{-11})
	121	0.832315918	922066.433	218.864	1.7376	(0.1416×10^{-9})
	181	0.863281732	420327.342	36.577	1.4254	(0.1765×10^{-8})
	241	0.885353369	240878.112	9.106	1.0978	(0.9600×10^{-8})
	301	0.902235495	156550.382	2.647	0.7661	(0.3494×10^{-7})
30	31	0.799549831	20331848.780	86361.176	2.0410	(0.2125×10^{-13})
	61	0.834661526	5355413.665	5419.713	1.8828	(0.7923×10^{-12})
	121	0.871544613	1390980.420	311.242	1.6380	(0.5034×10^{-10})
	181	0.893698241	630380.363	52.942	1.3757	(0.6076×10^{-9})
	241	0.909485263	359424.734	13.353	1.0789	(0.3328×10^{-8})
	301	0.921625760	232514.618	3.881	0.7561	(0.1227×10^{-7})
100	31	0.922148149	71065719.810	246029.875	1.6635	(0.5270×10^{-15})
	61	0.934517327	18489085.450	15926.483	1.6026	(0.2958×10^{-13})
	121	0.947181372	4734839.845	947.081	1.4643	(0.1824×10^{-11})
	181	0.954677942	2125739.828	165.370	1.2743	(0.2093×10^{-10})
	241	0.960012339	1203058.506	42.597	1.0282	(0.1159×10^{-9})
	301	0.964142825	773303.580	12.442	0.7288	(0.4360×10^{-9})

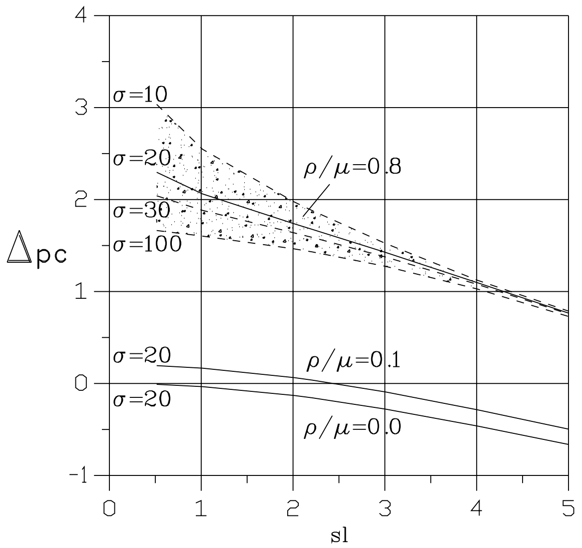

Fig. 7. Nonlinear-elastic model: post-critical coefficient Δ_{pc} vs. slenderness and constitutive coefficients.

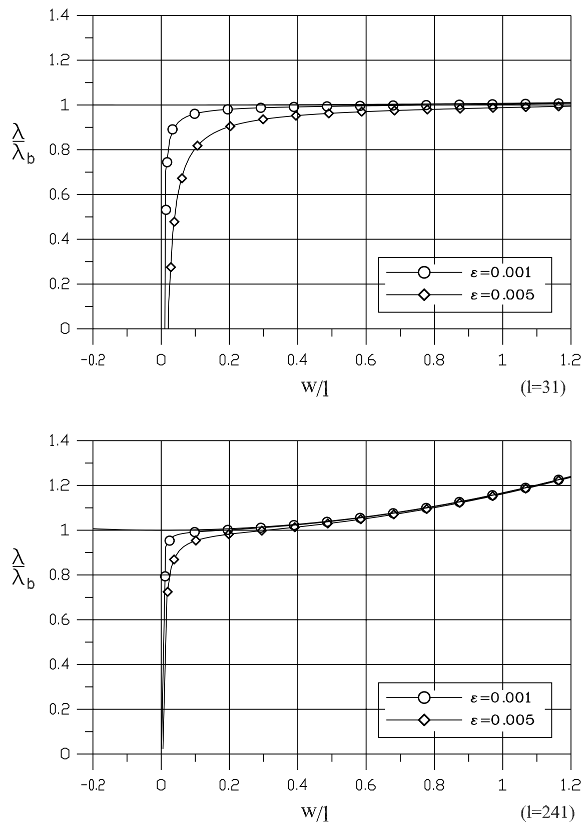

Fig. 8. Equilibrium paths of the nonlinear-elastic model.

values of the ratio ρ/μ, the values of the post-critical coefficient go back to the positive range, attaining a relevant stabilizing effect for $\rho/\mu \approx 1$.

The geometry of the element also affects the post-critical values. However, in a different way as to what was observed for the linear-elastic model, the higher values of the coefficient Δ_{pc} are attained for the less slender bearings.

For comparison, in Tables 2 and 3 the post-critical parameter $\tilde{\Delta}_{pc} = (\frac{1}{2} \times \frac{\ddot{\lambda}_b}{\lambda_b} \times l^2)_{eq}$ relative to the equivalent linear beam model in the sense of (34) are also reported: although the post-critical behavior of the nonlinear model is variable, the linear model is characterized by a post-critical behavior which is always stable but to a limited extent.

Finally, in the diagrams of Fig. 8 the curves of the equilibrium path for two typical cases of the observed behavior, stable (case $\rho/\mu = 0.8$) and unstable (case $\rho/\mu = 0.0$) are represented. For both cases the load-carrying capacity of imperfect structural elements shows significant differences from the critical load value of the perfect elements.

8. Conclusions

The paper has proposed and studied two different beam models for laminated elastomeric bearings. This study has been mainly oriented towards the evaluation of the critical and post-critical behavior of these structural models in the framework of a Koiter perturbation approach. For the linear-elastic model, the results of the analysis indicate that its post-critical behavior does not quantitatively affect the load-carrying capacity of the bearings (proving, for this model, the idea that the stability analysis can be resolved by only computing the critical load of the problem). On the contrary, the nonlinear-elastic beam model exhibits a strongly variegated post-critical behavior that certainly affects the load-carrying capacity of the structural element. Therefore, the stability analysis cannot be confined to the determination of the critical load but needs an exact evaluation of the post-critical behavior.

However the question of the accuracy of both models in predicting the behavior of real laminated elastomeric bearings is far from closed. In particular the problem of defining the constitutive coefficients (EA, GA, EJ) (μ, ρ, σ) on the basis of a homogeneization criterion that takes into account the real constructive geometry of laminated bearings is still open. In fact, the choice of that criterion is clearly essential to obtain an exact evaluation of the load-carrying capacity of the structural elements: indeed it is more true and delicate for the nonlinear-elastic model, where the criteria must take into account equivalent behavior conditions not only confined to the critical load but also able to cover the post-critical range. This theme is deferred to a future work.

Appendix A. Some kinematical relations

$$\delta\varepsilon = \varepsilon'\delta u = \delta u_{,s}\cos\theta - \delta w_{,s}\sin\theta - \delta\theta\gamma$$
$$\delta\gamma = \gamma'\delta u = \delta u_{,s}\sin\theta + \delta w_{,s}\cos\theta + \delta\theta(1+\varepsilon)$$
$$\delta\chi = \chi'\delta u = \delta\theta_{,s}$$

$$\varepsilon''\dot{u}\delta u = -\delta u_{,s}\dot{\theta}\sin\theta - \delta w_{,s}\dot{\theta}\cos\theta - \delta\theta(\gamma'\dot{u})$$
$$\gamma''\dot{u}\delta u = \delta u_{,s}\dot{\theta}\cos\theta - \delta w_{,s}\dot{\theta}\sin\theta + \delta\theta(\varepsilon'\dot{u})$$
$$\chi''\dot{u}\delta u = 0$$

$$\varepsilon'''\hat{u}\dot{u}\delta u = -\delta u_{,s}\dot{\theta}\hat{\theta}\cos\theta + \delta w_{,s}\dot{\theta}\hat{\theta}\sin\theta - \delta\theta(\gamma''\hat{u}\dot{u})$$
$$\gamma'''\hat{v}\dot{v}\delta u = -\delta u_{,s}\dot{\theta}\hat{\theta}\sin\theta - \delta w_{,s}\dot{\theta}\hat{\theta}\cos\theta + \delta\theta(\varepsilon''\hat{u}\dot{u})$$
$$\varepsilon''''\tilde{u}\hat{u}\dot{u}\delta u = \delta u_{,s}\dot{\theta}\hat{\theta}\tilde{\theta}\sin\theta + \delta w_{,s}\dot{\theta}\hat{\theta}\tilde{\theta}\cos\theta - \delta\theta(\gamma'''\tilde{u}\hat{u}\dot{u})$$

Along the fundamental path $(\varepsilon = \varepsilon_o, w = \theta = 0, \gamma = \chi = 0)$

$$\varepsilon'_o\delta u = \delta u_{,s}$$
$$\gamma'_o\delta u = \delta w_{,s} + \delta\theta(1+\varepsilon_o)$$

$$\varepsilon''_o\dot{u}\delta u = -\delta w_{,s}\dot{\theta} - \delta\theta\dot{w}_{,s} - \delta\theta\dot{\theta}(1+\varepsilon_o)$$
$$\gamma''_o\dot{u}\delta u = \delta u_{,s}\dot{\theta} + \delta\theta\dot{u}_{,s}$$

$$\varepsilon'''_o\hat{u}\dot{u}\delta u = -\delta u_{,s}\dot{\theta}\hat{\theta} - \delta\theta(\hat{u}_{,s}\dot{\theta} + \hat{\theta}\dot{u}_{,s})$$
$$\gamma'''_o\hat{u}\dot{u}\delta u = -\hat{\theta}\dot{\theta}\delta w_{,s} - \hat{w}_{,s}\dot{\theta}\delta\theta - \hat{\theta}\dot{w}_{,s}\delta\theta - \hat{\theta}\dot{\theta}\delta\theta(1+\varepsilon_o)$$

$$\varepsilon''''_o\tilde{u}\hat{u}\dot{u}\delta u = \delta w_{,s}\dot{\theta}\hat{\theta}\tilde{\theta} - \delta\theta(\gamma'''_o\tilde{u}\hat{u}\dot{u})$$

In the bifurcation configuration ($\dot{u}_{,s} = \ddot{w}_{,s} = \ddot{\theta} = 0$)

$$(\varepsilon_b'\dot{v}_b) = \dot{u}_{,s} = 0 \quad (\gamma_b'\dot{v}_b) = \dot{w}_{,s} + (1+\varepsilon_b)\dot{\theta}$$
$$(\varepsilon_b'\ddot{v}_b) = \ddot{u}_{,s} \quad (\gamma_b'\ddot{v}_b) = \ddot{w}_{,s} + (1+\varepsilon_b)\ddot{\theta} = 0$$
$$(\varepsilon_b''\dot{v}_b^2) = -2\dot{w}_{,s}\dot{\theta} - (1+\varepsilon_b)\dot{\theta}^2 \quad (\gamma_b''\dot{v}_b^2) = 2\dot{u}_{,s}\dot{\theta} = 0$$
$$(\varepsilon_b'''\dot{v}_b^3) = -3\dot{u}_{,s}\dot{\theta}\dot{\theta} = 0 \quad (\gamma_b'''\dot{v}_b^3) = -3\dot{\theta}^2\dot{w}_{,s} - (1+\varepsilon_b)\dot{\theta}^3$$
$$(\varepsilon_b''''\dot{v}_b^4) = 4\dot{\theta}^3\dot{w}_{,s} + (1+\varepsilon_b)\dot{\theta}^4 \quad (\gamma_b''\dot{v}_b\delta u) = \delta u_{,s}\dot{\theta} + \delta\theta\dot{u}_{,s} = \delta u_{,s}\dot{\theta}$$
$$(\varepsilon_b'''\dot{v}_b^2\delta u) = -2\dot{u}_{,s}\dot{\theta}\delta\theta - \delta u_{,s}\dot{\theta}^2 = -\delta u_{,s}\dot{\theta}^2 \quad (\gamma_b''\hat{u}\dot{v}_b) = \hat{u}_{,s}\dot{\theta} + \hat{\theta}\dot{u}_{,s} = \hat{u}_{,s}\dot{\theta}$$
$$(\varepsilon_b'''\hat{u}\dot{v}_b^2) = -2\dot{u}_{,s}\dot{\theta}\hat{\theta} - \hat{u}_{,s}\dot{\theta}^2 = -\hat{u}_{,s}\dot{\theta}^2 \quad (\chi_b'\dot{v}_b) = \dot{\theta}_{,s}$$
$$(\varepsilon_b''\ddot{v}_b^2) = -2\ddot{w}_{,s}\ddot{\theta} - (1+\varepsilon_b)\ddot{\theta}^2 = 0 \quad (\chi_b'\ddot{v}_b) = \ddot{\theta}_{,s} = 0$$
$$(\varepsilon_b''\hat{u}_b\dot{v}_b) = -\hat{w}_{,s}\dot{\theta} - \dot{w}_{,s}\hat{\theta} - (1+\varepsilon_b)\hat{\theta}\dot{\theta} = 0$$

Appendix B. Higher variation of the strain energy

Along the fundamental path ($T_o = M_o = 0$):

$$\Phi_o''\dot{u}\delta u = \int_l \Big\{(N_o'\dot{u})(\varepsilon_o'\delta u) + (T_o'\dot{u})(\gamma_o'\delta u) + (M_o'\dot{u})(\chi_o'\delta u) + N_o(\varepsilon_o''\dot{u}\delta u)\Big\}\mathrm{d}s$$

$$\begin{aligned}\Phi_o'''\hat{u}\dot{u}\delta u = \int_l \Big\{&(N_o''\hat{u}\dot{u})(\varepsilon_o'\delta u) + (N_o'\dot{u})(\varepsilon_o''\hat{u}\delta u) + (N_o'\hat{u})(\varepsilon''\dot{u}\delta u) + N_o(\varepsilon_o'''\hat{u}\dot{u}\delta u) + (T_o''\hat{u}\dot{u})(\gamma_o'\delta u)\\ &+ (T_o'\dot{u})(\gamma_o''\hat{u}\delta u) + (T_o'\hat{u})(\gamma_o''\dot{u}\delta u) + (M_o''\hat{u}\dot{u})(\chi_o'\delta u)\Big\}\end{aligned}$$

$$\begin{aligned}\Phi_o''''\tilde{u}\hat{u}\dot{u}\delta u = \int_l \Big\{&(N_o''\tilde{u}\hat{u})(\varepsilon_o''\dot{u}\delta u) + (N_o''\tilde{u}\dot{u})(\varepsilon_o''\hat{u}\delta u) + (N_o''\hat{u}\dot{u})(\varepsilon_o''\tilde{u}\delta u) + (N_o'\tilde{u})(\varepsilon_o'''\hat{u}\dot{u}\delta u) + (N_o'\hat{u})(\varepsilon_o'''\tilde{u}\dot{u}\delta u)\\ &+ (N_o'\dot{u})(\varepsilon_o'''\tilde{u}\hat{u}\delta u) + (N_o'''\tilde{u}\hat{u}\dot{u})(\varepsilon_o'\delta u) + N_o(\varepsilon_o''''\tilde{u}\hat{u}\dot{u}\delta u) + (T_o''\tilde{u}\hat{u})(\gamma_o''\dot{u}\delta u) + (T_o''\tilde{u}\dot{u})(\gamma_o''\hat{u}\delta u)\\ &+ (T_o''\hat{u}\dot{u})(\gamma_o''\tilde{u}\delta u) + (T_o'\tilde{u})(\gamma_o'''\hat{u}\dot{u}\delta u) + (T_o'\hat{u})(\gamma_o'''\tilde{u}\dot{u}\delta u) + (T_o'\dot{u})(\gamma_o'''\tilde{u}\hat{u}\delta u) + (T_o'''\tilde{u}\hat{u}\dot{u})(\gamma_o'\delta u)\\ &+ (M_o'''\tilde{u}\hat{u}\dot{u})(\chi'\delta u)\Big\}\mathrm{d}s\end{aligned}$$

In the bifurcation configuration ($\varepsilon_b'\dot{v}_b = \gamma_b''\dot{v}_b^2 = \varepsilon_b'''\dot{v}_b^3 = \gamma_b'\ddot{v}_b = \chi_b'\ddot{v}_b = \varepsilon_b''\ddot{v}_b^2 = 0$)

$$\Phi_b''\ddot{v}_b^2 = \int_l \Big\{(N_b'\ddot{v}_b)(\varepsilon_b'\ddot{v}_b)\Big\}\mathrm{d}s$$

$$\begin{aligned}\Phi_b'''\dot{v}_b^2\delta u = \int_l \Big\{&(N_b''\dot{v}_b^2)(\varepsilon_b'\delta u) + 2(N_b'\dot{v}_b)(\varepsilon''\dot{v}_b\delta u) + N_b(\varepsilon_b'''\dot{v}_b^2\delta u) + (T_b''\dot{v}_b^2)(\gamma_b'\delta u) + 2(T_b'\dot{v}_b)(\gamma_b''\dot{v}_b\delta u)\\ &+ (M_b''\dot{v}_b^2)(\chi_b'\delta u)\Big\}\mathrm{d}s\end{aligned}$$

$$\Phi_b'''\dot{v}_b^3 = \int_l \Big\{2(N_b'\dot{v}_b)(\varepsilon_b''\dot{v}_b^2) + (T_b''\dot{v}_b^2)(\gamma_b'\dot{v}_b) + (M_b''\dot{v}_b^2)(\chi_b'\dot{v}_b)\Big\}\mathrm{d}s$$

$$\Phi_b'''\hat{u}_b\dot{v}_b^2 = \int_l \Big\{(N_b'\hat{u})(\varepsilon''\dot{v}_b^2) + N_b(\varepsilon_b'''\hat{u}\dot{v}_b^2) + (T_b''\hat{u}\dot{v}_b)(\gamma_b'\dot{v}_b) + (T_b'\dot{v}_b)(\gamma_b''\hat{u}\dot{v}_b) + (M_b''\hat{u}\dot{v}_b)(\chi_b'\dot{v}_b)\Big\}\mathrm{d}s$$

$$\Phi_b''''\dot{v}_b^4 = \int_l \Big\{3(N_b''\dot{v}_b^2)(\varepsilon_b''\dot{v}_b^2) + N_b(\varepsilon_o''''\dot{v}_b^4) + 3(T_b'\dot{v}_b)(\gamma_b'''\dot{v}_b^3) + (T_b'''\dot{v}_b^3)(\gamma_b'\dot{v}_b) + (M_b'''\dot{v}_b^3)(\chi_b'\dot{v}_b)\Big\}\mathrm{d}s$$

Appendix C. The nonlinear-elastic model case

C.1. Constitutive relations

Set $k = 1 + \varepsilon$ and using the following functions:

$$\alpha[x] = \mu\left[1 + (1+\sigma)x^{-(2+\sigma)}\right] + \rho[3x^2 - 1]$$
$$\beta[x] = \mu + \rho(x^2 - 1)$$
$$\eta[x] = \mu x^{-(1+\sigma)}$$

the nonlinear elastic constitutive model is expressed by

$$N = \int_A \frac{\partial}{\partial \varepsilon}\varphi[\varepsilon,\gamma,\chi]\,\mathrm{d}A = \int_A \{(k+z\chi)\beta[k+z\chi] - \eta[k+z\chi]\}\mathrm{d}A + A\rho\gamma^2 k$$
$$T = \int_A \frac{\partial}{\partial \gamma}\varphi[\varepsilon,\gamma,\chi]\,\mathrm{d}A = \int_A \{\gamma\beta[k+z\chi]\}\mathrm{d}A + A\rho\gamma^3$$
$$M = \int_A \frac{\partial}{\partial \chi}\varphi[\varepsilon,\gamma,\chi]\,\mathrm{d}A = \int_A \{z(k+z\chi)\beta[k+z\chi] - z\eta[k+z\chi]\}\mathrm{d}A + J\rho\gamma^2\chi$$

Along the fundamental path ($\gamma = \chi = 0$)

$$N_\mathrm{o} = Ak\beta[k] - A\eta[k], \quad T_\mathrm{o} = 0, \quad M_\mathrm{o} = 0$$

C.2. First variation of the constitutive relations along the fundamental path

$$\dot{N}_\mathrm{o} = N'_\mathrm{o}\dot{u} = A\alpha[k](\varepsilon'_\mathrm{o}\dot{u}) = A\alpha[k]\dot{u}_{,s}$$
$$\dot{T}_\mathrm{o} = T'_\mathrm{o}\dot{u} = A\beta[k](\gamma'_\mathrm{o}\dot{u}) = A\beta[k]\dot{w}_{,s} + A\eta[k]\dot{\theta} + N_\mathrm{o}\dot{\theta}$$
$$\dot{M}_\mathrm{o} = M'_\mathrm{o}\dot{u} = J\alpha[k](\chi'_\mathrm{o}\dot{u}) = J\alpha[k]\dot{\theta}_{,s}$$

C.3. Some quantities in the bifurcation configuration

Set $\alpha_\mathrm{b} = \alpha[k_\mathrm{b}]$, $\beta_\mathrm{b} = \beta[k_\mathrm{b}]$, $\eta_\mathrm{b} = \eta[k_\mathrm{b}]$, $\omega_\mathrm{b} = \frac{\partial\alpha}{\partial x}[k_\mathrm{b}]$ $\tau_\mathrm{b} = \frac{\partial^2\alpha}{\partial x^2}[k_\mathrm{b}]$

C.3.1. Strain

$$(\gamma'_\mathrm{b}\dot{v}_\mathrm{b}) = \dot{w}_{,s} + k_\mathrm{b}\dot{\theta} = \left(-\frac{\eta_\mathrm{b}}{\beta_\mathrm{b}} + k_\mathrm{b}\right)\dot{\theta} = \frac{N_\mathrm{b}}{A\beta_\mathrm{b}}\dot{\theta}$$

$$(\varepsilon''_\mathrm{b}\dot{v}^2_\mathrm{b}) = -2\dot{w}_{,s}\dot{\theta} - k_\mathrm{b}\dot{\theta}^2 = \left(+2\frac{\eta_\mathrm{b}}{\beta_\mathrm{b}} - k_\mathrm{b}\right)\dot{\theta}^2 = \left(k_\mathrm{b} - 2\frac{N_\mathrm{b}}{A\beta_\mathrm{b}}\right)\dot{\theta}^2$$

$$(\gamma'''_\mathrm{b}\dot{v}^3_\mathrm{b}) = -3\dot{\theta}^2\dot{w}_{,s} - k_\mathrm{b}\dot{\theta}^3 = \left(+3\frac{\eta_\mathrm{b}}{\beta_\mathrm{b}} - k_\mathrm{b}\right)\dot{\theta}^3 = \left(2k_\mathrm{b} - 3\frac{N_\mathrm{b}}{A\beta_\mathrm{b}}\right)\dot{\theta}^3$$

$$(\varepsilon''''_\mathrm{b}\dot{v}^4_\mathrm{b}) = 4\dot{\theta}^3\dot{w}_{,s} + k_\mathrm{b}\dot{\theta}^4 = \left(-4\frac{\eta_\mathrm{b}}{\beta_\mathrm{b}} + k_\mathrm{b}\right)\dot{\theta}^4 = \left(-3k_\mathrm{b} + 4\frac{N_\mathrm{b}}{A\beta_\mathrm{b}}\right)\dot{\theta}^4$$

C.3.2. Tension

$$
\begin{aligned}
(N_b'\dot{v}_b) &= \dot{N} = A\alpha_b\dot{u}_{,s} = 0\\
(N_b'\hat{u}_b) &= \hat{N}_b = A\alpha_b\hat{u}_{,s} = \frac{\partial}{\partial\lambda}N[\lambda] = -1\\
(T_b'\dot{v}_b) &= \dot{T} = N_b\dot{\theta}\\
(T_b''\dot{v}_b^2) &= A\beta_b(\gamma_b''\dot{v}_b^2) + A4\rho k_b(\gamma_b'\dot{v}_b)(\varepsilon_b'\dot{v}_b) = 0\\
(T_b''\hat{u}\dot{v}_b) &= A\beta_b(\gamma_b''\hat{u}\dot{v}_b) + \rho A\Big(2k_b(\gamma_b'\hat{u}_b)(\varepsilon_b'\dot{v}_b) + 2k_b(\varepsilon_b'\hat{u}_b)(\gamma_b'\dot{v}_b)\Big)\\
&= \hat{u}_{,s}\dot{\theta}\left(A\beta_b + 2\rho k_b\frac{N_b}{\beta_b}\right)\\
(M_b''\dot{v}_b^2) &= J\omega_b 2(\varepsilon'\dot{v}_b)\dot{\theta}_{,s} = 0\\
(M_b''\hat{u}\dot{v}_b) &= J\omega_b\Big((\varepsilon'\hat{u}_b)\dot{\theta}_{,s} + (\varepsilon'\dot{v}_b)\hat{\theta}_{,s}\Big)\\
&= J\omega_b\hat{u}_{,s}\dot{\theta}_{,s}\\
(N_b''\dot{v}_b^2) &= A\alpha_b(\varepsilon_b''\dot{v}_b^2) + A2\rho k_b(\gamma_b'\dot{v}_b)^2 + J\omega_b(\dot{\theta}_{,s})^2\\
&= A\left(\alpha_b k_b - 2\alpha_b\frac{N_b}{A\beta_b} + 2\rho k_b\frac{N_b^2}{A^2\beta_b^2}\right)\dot{\theta}^2 + J\omega_b\dot{\theta}_{,s}^2\\
T_b'''\dot{v}_b^3 &= \beta_b A(\gamma_b'''\dot{v}_b^3) + \rho A\Big[6(\gamma_b'\dot{v}_b)(\varepsilon_b''\dot{v}_b^2)\Big]k_b + \rho A\Big[6(\gamma_b'\dot{v}_b)^3\Big] + \rho J\Big[6(\gamma_b'\dot{v}_b)\dot{\theta}_{,s}\dot{\theta}_{,s}\Big]\\
&= A\left[2k_b\beta_b + (\rho 6k_b^2 - 3\beta_b)\frac{N_b}{A\beta_b} - 12\rho k_b\frac{N_b^2}{A^2\beta_b^2} + 6\rho\frac{N_b^3}{A^3\beta_b^3}\right]\dot{\theta}^3 + J6\rho\frac{N_b}{A\beta_b}\dot{\theta}\dot{\theta}_{,s}^2\\
M_b'''\dot{v}_b^3 &= 3\omega_b J\dot{\theta}_{,s}(\varepsilon''\dot{v}_b^2) + \tau_b J_4\dot{\theta}_{,s}^3\\
&= J\left[3k_b\omega_b - 6\omega_b\frac{N_b}{A\beta_b}\right]\dot{\theta}_{,s}\dot{\theta}^2 + J_4\tau_b\dot{\theta}_{,s}^3
\end{aligned}
$$

C.3.3. Energy

$$\Phi_b'''\dot{v}_b^3 = 0$$

$$\Phi_b'''\dot{v}_b^2\delta u = A\left(\alpha_b k_b + (\beta_b - 2\alpha_b)\frac{N_b}{A\beta_b} + 2\rho k_b\frac{N_b^2}{A^2\beta_b^2}\right)\int_l\Big\{\dot{\theta}^2\delta u_{,s}\Big\}\mathrm{d}s + J\omega_b\int_l\Big\{\dot{\theta}_{,s}^2\delta u_{,s}\Big\}\mathrm{d}s$$

$$
\begin{aligned}
\Phi_b'''\hat{u}_b\dot{v}_b^2 &= A\left(\alpha_b k_b + (\beta_b - 2\alpha_b)\frac{N_b}{A\beta_b} + 2\rho k_b\frac{N_b^2}{A^2\beta_b^2}\right)\int_l\Big\{\dot{\theta}^2\hat{u}_{,s}\Big\}\mathrm{d}s + J\omega_b\int_l\Big\{\dot{\theta}_{,s}^2\hat{u}_{,s}\Big\}\mathrm{d}s\\
&= \left(\frac{1}{2}l\right)\left(\frac{\pi}{2l}\frac{\beta_b}{\eta_b}\right)^2\left(\frac{-1}{\alpha_b}\right)\left(\alpha_b k_b + \left(\beta_b - 2\alpha_b - \frac{\omega_b\eta_b}{\alpha_b}\right)\frac{N_b}{A\beta_b} + 2\rho k_b\frac{N_b^2}{A^2\beta_b^2}\right)
\end{aligned}
$$

$$\Phi_b'''' \dot{v}_b^4 - 3\Phi_b'' \ddot{v}_b^2 = \left(\frac{\pi}{2l}\frac{\beta_b}{\eta_b}\right)^4 \frac{3}{8} lA \left\{ \left(\frac{J_4}{A}\frac{\pi^4}{l^4}\tau_b\right) + \frac{N_b}{A\beta_b}\frac{\eta_b}{\alpha_b}\left(-k_b\beta_b\frac{\alpha_b}{\eta_b}\right) + \frac{N_b^2}{A^2\beta_b^2}\left(4\beta_b - 3\frac{\beta_b^2}{\alpha_b} + 2\frac{\omega_b\beta_b\eta_b}{\alpha_b^2} - 3\frac{\omega_b^2\eta_b^2}{\alpha_b^3}\right) + \frac{N_b^3}{A^3\beta_b^3}\frac{\eta_b}{\alpha_b} \times \left(-2\rho + 4\rho k_b\frac{\omega_b}{\alpha_b} - 12\frac{\rho k_b\beta_b}{\eta_b}\right) + \frac{N_b^4}{A^4\beta_b^4}\left(6\rho - 12\frac{\rho^2 k_b^2}{\alpha_b}\right)\right\}$$

C.3.4. Post-critical curvature expression

$$\ddot{\lambda}_b = -\frac{\Phi_b'''' \dot{v}_b^4 - 3\Phi_b'' \ddot{v}_b^2}{3\Phi_b''' \hat{u}\dot{v}_b^2} = \frac{1}{4}\left(\frac{\pi}{2l}\frac{\beta_b}{\eta_b}\right)^2 A \frac{\left(C_0 + \frac{N_b}{A\beta_b}C_1 + \frac{N_b^2}{A^2\beta_b^2}C_2 + \frac{N_b^3}{A^3\beta_b^3}C_3 + \frac{N_b^4}{A^4\beta_b^4}C_4\right)}{D_0 + \frac{N_b}{A\beta_b}D_1 + \frac{N_b^2}{A^2\beta_b^2}D_2}$$

with

$$C_0 = \left(\frac{J_4}{A}\frac{\pi^4}{l^4}\alpha_b\tau_b\right) \quad D_0 = (\alpha_b k_b)$$

$$C_1 = (-k_b\beta_b\alpha_b) \quad D_1 = \left(\beta_b - 2\alpha_b - \frac{\omega_b\eta_b}{\alpha_b}\right)$$

$$C_2 = \left(4\alpha_b\beta_b - 3\beta_b^2 + 2\frac{\omega_b\beta_b\eta_b}{\alpha_b} - 3\frac{\omega_b^2\eta_b^2}{\alpha_b^2}\right) \quad D_2 = (2\rho k_b)$$

$$C_3 = \left(-2\rho\eta_b + 4\rho k_b\frac{\omega_b\eta_b}{\alpha_b} - 12\rho k_b\beta_b\right) \quad C_4 = \left(6\rho\alpha_b - 12\rho^2 k_b^2\right)$$

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